The History and Development of Logarithms

by

Meaghan Whitley Bennett, BA

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Approved by
Supervising Committee:

Efraim P. Armendariz

Mark L. Daniels

Abstract

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Meaghan Whitley Bennett, MA

The University of Texas at Austin, 2009

Supervisor: Efraim Armendariz

This paper outlines the evolution of the logarithm from the days of Archimedes to the logarithm now used in modern mathematics. Each type of logarithm developed had its particular usefulness. The Archimedean logarithm helped astronomers by drastically shortening the time it took to multiply large numbers, while Napier's logarithm could be used as a tool to solve velocity problems. With the discovery of the number *e*, the natural logarithm was developed. Due to the frequent use of *e*, many of the properties of logarithms were defined to work nicely for the natural logarithm to make calculations easier. This paper will explain the proofs and connections of such properties in a way that could be presented in a calculus class.

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Chapter 1: *Introduction*

The topic of logarithms appear in multiple high school courses. Logarithms are first introduced in Algebra II, expanded upon in Pre-Calculus, and are applied in Calculus. Many high school students still struggle with logarithms when taught in Calculus. The high school curriculum focuses heavily on stating the rules of logarithms, how to convert between logarithmic and exponential form, and how to use them to solve basic algebra equations. The concept of what a logarithm is and how it was developed is completely passed over. Students are asked to blindly accept the fact that these functions exist and to learn their properties and rules to solve problems. In addition, teachers are asked to introduce the natural logarithm and the concept of *e* to students with no explanation other than the fact that it is a logarithm with a special base. When logarithms were first introduced into the mathematical world, they were not the logarithms we know now. Archimedes developed a very basic concept of logarithms with his creation of the *order* of a number. Napier and Briggs learned of the tedious calculations being done by astronomers using trigonometric identities and developed a more recognizable concept of logarithms. Euler later developed definitions for many of our commonly used terms relating to logarithms as well as certain relations between logarithms, including natural logarithms. By exploring these developments and concepts more in depth, teachers can construct a more meaningful curriculum for our high school students.

Chapter 2: *The History and Development of Logarithms*

Logarithms have not always been defined in the manner now commonly used. One of the earliest forms of logarithms was seen in the time of Archimedes. The mathematician defined the "order" of a number to be "equivalent to the exponent where the base is 100,000,000" (Pierce, 1977, p.22). Archimedes also noticed that by relating the geometric progression of numbers to an arithmetic progression one would be able to simplify multiplication problems into addition problems (Bruce, 2000). This is a concept that Napier later expanded upon.

Tycho Brahe was an astronomer who was working to disprove the Copernican theory of planetary motion (Pierce, 1977). Brahe's efforts required many computations which used a method called *prosthaphaeresis*. This method used trigonometric identities to find the products of large numbers. For example, the product of 2250 and 1219 would be found by computing

$$(\cos A)(\cos B) = \frac{\cos(A+B) + \cos(A-B)}{2}.$$

By allowing $\cos A = .2250$ and $\cos B = .1219$ the identity yields

$$(.2250)(.1219) = \frac{\cos(77° + 83°) + \cos(77° - 83°)}{2} = .0274.$$

Thus,

$$(2250)(1229) = 2{,}740{,}000.$$

By chance, a man by the name of Dr. John Craig, physician to James VI, learned of the method being used by Brahe and communicated it to John Napier upon his return to Scotland. After being told of the prosthaphaeresis method, Napier was said to have doubled his efforts to find a computational aid (Pierce, 1977). Napier began by expanding on Archimedes discovery of the relationship between the geometric and arithmetic progressions of numbers. An example of this relationship is seen in Table 1.

Table 1: Napier's Algebraic versus Geometric Progression

$$2 \leftrightarrow 3$$

$$4 \leftrightarrow 9$$

$$6 \leftrightarrow 27$$

$$8 \leftrightarrow 81$$

$$10 \leftrightarrow 243, \text{etc.} \qquad \text{(Pierce, 1977).}$$

Each term in the arithmetic progression from column one is the log of the corresponding term in the geometric progression from column two. These progressions can be used to solve multiplication problems such as (9) (27) by stating

$$\log(9)\,(27) = \log(9) + \log(27) = 4 + 6 = 10.$$

Since 10 is the log of 243, then (9) (27) = 243 (Pierce, 1977). It is interesting to note that the property of logarithms

$$\log(ab) = \log(a) + \log(b)$$

was used to solve these multiplication problems even though it was not formally proven until Euler's definition of logarithm was developed. Napier's problem at this point was that he knew, "to be useful, one needs the numbers in this progression to be close together, so that *any* number can be represented approximately" (Bruce, 2000, p. 149). Napier overcame this problem by setting up a geometric sequence with a very small common ratio. The mathematician then chose to look at it as a distance versus time problem. Assume that a point travels $\left(1 - \frac{1}{10^7}\right)$ of a unit from the starting point for each one unit of time (Pierce, 1977). This creates a geometric sequence with a common ration of .9999999 between terms and an arithmetic progression with a common difference of one, which allowed Napier to calculate log values to an acceptable place value. Each time, t is the logarithm of the corresponding distance the point traveled in that time (Pierce, 1977). Napier used the method to complete his "Table of Radicals," which he then named logarithms, from the Greek words *logos* and *arithmos*, meaning ratio and number respectively (Pierce, 1977). The development of Napier's logarithm can be represented graphically as well. Let there be two points, G and H, traveling along a ray and a line segment respectively. Point G travels a constant distance, b, at each time interval. The distance point H travels each interval is increased by a multiple of a each time.

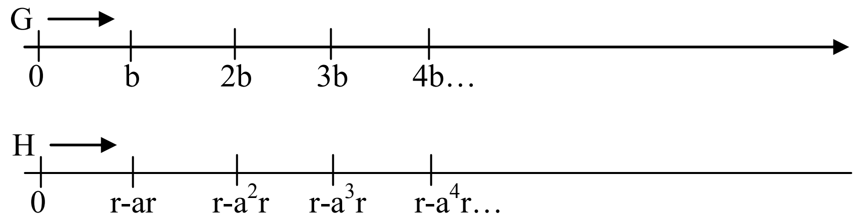

Figure 1: Napier's arithmetic versus geometric progressions.

Napier's logarithm, as visualized in Figure 1, can be viewed in calculus as an "instantaneous velocity" problem (Villarreal-Calderon, 2008). Let *r* be the entire distance *H* has to travel, and *x* be the distance *H* has left to travel, so that $(r - x)$ represents the distance *H* has traveled. The velocity of point H, V_H, can be represented by

$$V_H = \frac{d(r - x)}{dt} = \frac{dr}{dt} - \frac{dx}{dt}.$$

The velocity of *H* is proportional to the distance yet to travel. Thus,

$$\frac{dr}{dt} - \frac{dx}{dt} = kx$$

$$0 - \frac{dx}{dt} = kx, \text{so} - \frac{dx}{dt} = kx.$$

where k is any constant. Consider V_H when $k = 1,$ and note that $\frac{dr}{dt} = 0$ since r is a constant.

$$V_H = -\frac{dx}{dt} = (1)x = x.$$

$$\text{If} - \frac{dx}{dt} = x, \text{then } dx = -xdt$$

$$-\mathrm{dt} = \frac{\mathrm{dx}}{\mathrm{x}}$$

$$-\mathrm{dt} = \frac{1}{\mathrm{x}}\mathrm{dx}.$$

Thus, by integrating both sides

$$\int -\mathrm{dt} = \int \frac{1}{\mathrm{x}}\mathrm{dx}$$

$$-\int \mathrm{dt} = \int \frac{1}{\mathrm{x}}\mathrm{dx} - \mathrm{t} = \ln \mathrm{x} + \mathrm{c}.$$

At $t = 0$, points *G* and *H* have the same velocity, so $\mathrm{x} = \mathrm{r}$. This yields the following:

$$0 = \ln\mathrm{r} + \mathrm{c}$$

$$-\ln\mathrm{r} = \mathrm{c},$$

$$-\mathrm{t} = \ln\mathrm{x} - \ln\mathrm{r}$$

$$\mathrm{t} = \ln\mathrm{r} - \ln\mathrm{x}$$

$$\mathrm{t} = \ln\left(\frac{\mathrm{r}}{\mathrm{x}}\right) \quad (1)$$

Point *G* changes arithmetically and therefore travels at a constant velocity represented by $\frac{\mathrm{dy}}{\mathrm{dt}}$. The initial conditions previously stated were that $\mathrm{t} = 0$, $\mathrm{x} = \mathrm{r}$, and $\frac{\mathrm{dx}}{\mathrm{dt}} = \frac{\mathrm{dy}}{\mathrm{dt}}$ at that time. Based on those conditions, if follows that

$$\frac{\mathrm{dy}}{\mathrm{dt}} = \frac{\mathrm{dx}}{\mathrm{dt}} = \mathrm{r}.$$

By the same process as above,

$$\frac{\mathrm{dy}}{\mathrm{dt}} = \mathrm{r}$$

$$\mathrm{dy} = \mathrm{rdt}$$

$$\int dy = \int rdt$$

$$y = rt \tag{2}$$

The results in (1) and (2) can be combined to yield $y = r\ln\left(\frac{r}{x}\right)$. Napier assigned *r* to be 10^7, so if Napier's logarithm was set to the current definition of $\log x = y$, then in Napier's time,

$$\log x = 10^7 \ln\left(\frac{10^7}{x}\right).$$

It wasn't until late in his career, when Napier collaborated with Henry Briggs, that the logarithm tables were reconstructed using the common base of ten. Converting Napier's logs to a common base of ten allowed certain calculations to be much more convenient. The base ten allowed

$$\log(1) = 0,$$

instead of 10,000,000 by Napier's method (Villarreal-Calderon, 2008). Once such tables were created, it was said logarithms, "literally lengthened the life spans of astronomers, who had formerly been sorely bent and often broken early by the masses of calculations their art required" (Pierce, 1977, p. 26). Having a common base also allowed for many modern properties of logs to be applied such as

$$\log(xy) = \log(x) + \log(y). \tag{3}$$

This property and others will be proven formally below. Napier died before the new logarithm tables were complete, so Briggs completed and published them (Villarreal-

Calderon, 2008). He did calculations to find the logarithms of prime numbers first, and then used (3) to fill in the logarithm of many non-prime numbers. For example,

$$\log(10) = \log(2 * 5) = \log(2) + \log(5),$$

which are both prime numbers whose log values had previously been calculated (Villarreal-Calderon, 2008).

Many years later, Mercator named logarithms with base *e natural logarithms*. He observed that

$$\int_0^x \frac{1}{1+t}\,dt = \ln(1+x)$$

Thus,

$$\ln(1+x) = \frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

This was named the Mercator Series, and any logarithm that could be found by this series was called a "natural" logarithm (Pierce, 1977). The general form of this statement will be proven later in the paper. However, it was John Speidell who was known for modifying the Naperian logarithms to create what are now called natural logarithms. The only noticeable difference between the current natural logarithm and the one Speidell created was that Speidell did not use decimal places (Villarreal-Calderon, 2008). The number *e* shows up so often in nature, that it was found that a logarithm with base *e* would be quite useful. The number *e* was defined to be, "the limit of (1+1/n)^n as *n* approaches infinity" (pg. 340). Based on that definition, the natural logarithm is actually the easiest of all logarithms to derive (Villarreal-Calderon, 2008, p. 340). It can be shown that

$$\text{If } y = \ln(x), \text{then } \frac{d}{dx}[\ln(x)] = \frac{1}{x} \tag{4}$$

This statement could not be proven until Leonhard Euler introduced the connection between logarithms and exponential equations.

Euler was the first to define logarithms in terms of exponents by stating

$$\log_x y = z \text{ is true when } x^z = y \tag{*}$$

With this definition, it was now possible to prove (4) as well as all of the logarithmic properties used today.

To prove (4), begin by letting $y = \log_w x$ and choosing $w = e$. Based on Euler's definition if follows that

$$e^y = x.$$

Taking the derivative of each side with respect to *x* yields,

$$e^y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{e^y} = e^{-y}.$$

By substitution,

$$\frac{dy}{dx} = e^{-\ln(x)} = e^{\ln(x^{-1})} = x^{-1} = \frac{1}{x}.$$

It has only been shown that the derivative of the natural logarithm is $\frac{1}{x}$, but does the derivative hold true for any base? Begin with $y = \ln(x)$,

$$f'(x) = \lim_{h\to 0} \frac{\ln(x+h) - \ln(x)}{h} = \lim_{h\to 0} \frac{\ln\left(\frac{x+h}{x}\right)}{h} = \lim_{h\to 0} \frac{1}{h} \ln\left(\frac{x+h}{x}\right)$$

$$= \lim_{h \to 0} \ln \left(\frac{x+h}{x}\right)^{\frac{1}{h}}$$

Let $y = \frac{h}{x}$ and $h = xy$, then

$$f'(x) = \lim_{y \to 0} \left[\log_b (1+y)^{\frac{1}{xy}}\right] = \frac{1}{x} \lim_{y \to 0} \left[\log_b (1+y)^{\frac{1}{y}}\right]$$

$$= \frac{1}{x} \log_b \left[\lim_{y \to 0} (1+y)^{\frac{1}{y}}\right].$$

To see if the derivative of $\log_b x = \frac{1}{x}$, set

$$\frac{1}{x} = \frac{1}{x} \log_b \left[\lim_{y \to 0} (1+y)^{\frac{1}{y}}\right].$$

Then

$$\log_b \left[\lim_{y \to 0} (1+y)^{\frac{1}{y}}\right] = 1.$$

By substituting and using the definition of *e*

$$\log_b [e] = 1.$$

which will only be true if the base of the logarithm is *e*. This can be used as a stronger argument for the convenience of using the natural logarithm in calculus.

The basic logarithm properties, such as (3), can also be proven using the conversions between exponential and logarithmic form provided by Euler in (*). To prove the general addition property of logarithms, such as (3), let $\log_w a = x$ and $\log_w b = y$. Then

$$a = w^x \text{ and } b = w^y, \text{thus } ab = w^x w^y = w^{x+y}.$$

If $ab = w^{x+y}$, by definition, the logarithm of both sides yields

$$\log_w(ab) = \log_w w^{x+y} = x + y.$$

Thus, by substitution,

$$\log_w(ab) = \log_w a + \log_w b.$$

By a similar argument,

$$\log_w\left(\frac{a}{b}\right) = \log_w a - \log_w b \text{ and } \log_w a^x = x \log_w a.$$

Without a solid understanding of the definition of *e*, it is not always apparent to students that the statement

$$\int_1^r \frac{1}{x}\,dx = \log_e r\,,$$

implies the base of the logarithm must be *e*. It would also be true that

$$\frac{d}{dx}[\ln(x)] = \frac{1}{x}$$

only holds if the base of the logarithm is *e*, thus illustrating why the natural logarithm is the preferred logarithm in calculus. To assist in the understanding, Schmalz (1990) looked at the area under the curve $\frac{1}{x}$ bounded by the x-axis over the interval [1, r]. This area is represented by

$$\int_1^r \frac{1}{x}\,dx.$$

This integral statement says for area, *A*,

$$A = \ln(r) - \ln(1) = \ln(r).$$

Another method Schmalz used to evaluate the area was to divide the area into small trapezoids. The x-axis was partitioned by positive integer powers of *b* starting at $x = 1$

and ending at $b^n = r$. The area of each trapezoid was denoted by $T(a, b)$, and the total area under the curve is denoted by $A(r)$.

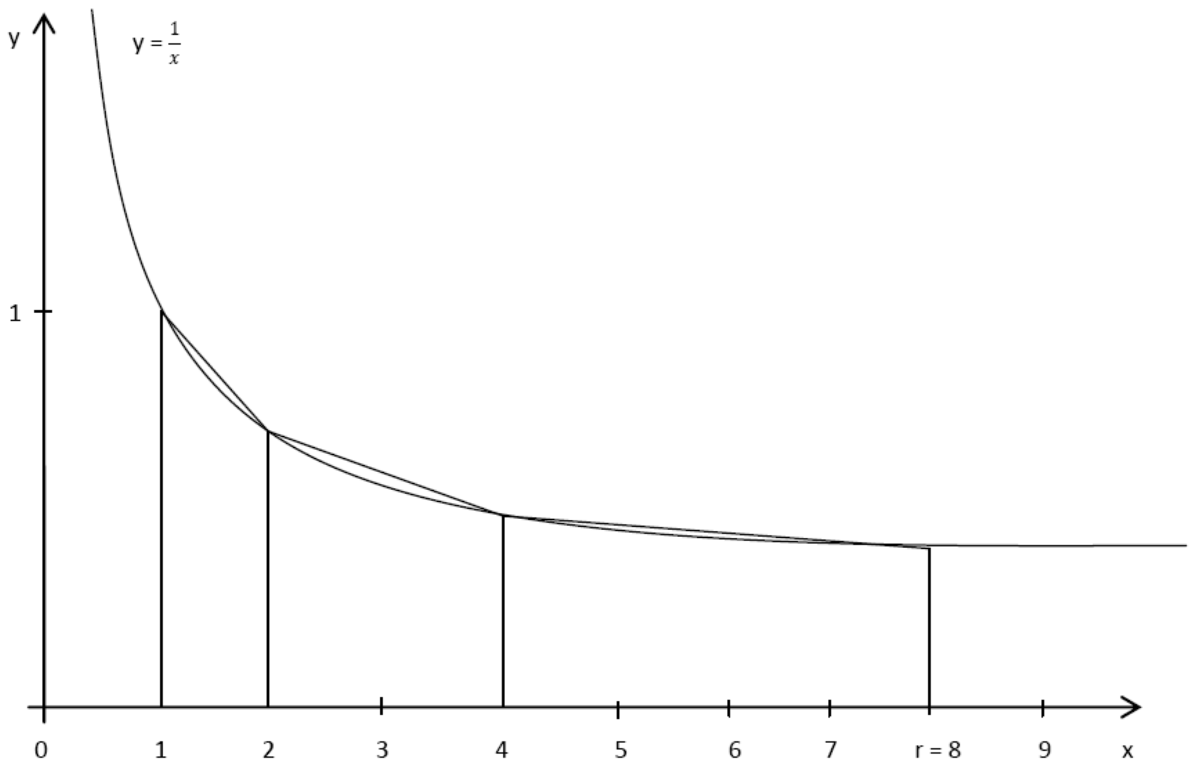

Figure 2: Graph of $f(x) = \frac{1}{x}$ divided into trapezoids using b = 2.

If $T(1,b)$ represents the area of a trapezoid, then the general formula for the area of any trapezoid bounded by b^m and b^{m+1} can be found by the following derivation.

The basic formula for the area of a trapezoid is $A = h\left(\frac{b_1+b_2}{2}\right)$, so

$$T(b^m, b^{m+1}) = (b^{m+1} - b^m)\left(\frac{\frac{1}{b^{m+1}} + \frac{1}{b^m}}{2}\right)$$

$$= (b^{m+1} - b^m)\left(\frac{b^m + b^{m+1}}{2b^{m+1}b^m}\right)$$

$$= b^m(b-1)\left(\frac{b+1}{2b^{m+1}}\right) = \frac{b^m(b-1)(b+1)}{2b^{m+1}} = \frac{b^m(b^2-1)}{2b^{m+1}}.$$

Factoring out a b^m yields

$$\frac{b^2-1}{2b}.$$

This shows that on the interval [1, r], the area of each trapezoid is the same, regardless of the bounds chosen. Thus

$$T(1, b) = T(b^m, b^{m+1}) = \frac{b^2-1}{2b}$$

If $A(r)$is the sum of *n* trapezoids, then

$$A(r) \approx n\left(\frac{b^2-1}{2b}\right)$$

and by substitution,

$$A(r) \approx \log_b r\left(\frac{b^2-1}{2b}\right).$$

As a result, if $T(1, b) = 1$, then the area under the curve would simply be $\log_b r$, which makes calculations very simple. If students let b = 2, then

$$T(1, b) = \frac{3}{4} < 1,$$

and if they let b = 3, then

$$T(1, b) = \frac{4}{3} > 1,$$

thereby allowing students to discover that the desired value for *b* would be between two and three. Therefore, "when the mysterious number *e* appears, it will be expected" (Schmalz, 1990). It is clear why using natural logarithms is preferred in Calculus.

With this knowledge, the basic properties of logarithms such as (3) can be proven using Calculus.

Since

$$\int_1^r \frac{1}{x}\,dx = \log(r) \text{ for } r > 0,$$

then,

$$\ln(ab) = \int_1^{ab} \frac{dx}{x} = \int_1^{a} \frac{dx}{x} + \int_a^{ab} \frac{dx}{x}.$$

Let $x = au$. Then,

$$= \ln(a) + \int_1^b \frac{du}{u} = \ln(a) + \ln(b). \qquad \text{(Gant, 1946)}$$

Even though the natural logarithm is used for its convenience, logarithms can be used with any base. Most applications are with real number bases, but logarithms can also have complex numbers as bases. One of the earlier noted applications of complex bases was in the famous debate between Bernoulli and Euler about whether or not the logarithm of a negative number existed. Was $\log(-x) = (x)$? The only solution to this dilemma involves the consideration of logarithms with complex bases. When dealing with complex bases, the properties of logarithms still hold, but the uniqueness of the solutions does not. When using real numbers for bases, there is one answer to each logarithm problem. For example, for $\log_b 5 = x$, there is only one x-value that will yield $b^x = 5$ for all real bases, *b*. When a base becomes complex, there are infinite solutions to every logarithm (Huestis, 1995). Let *z* be a complex number such that $z = x + iy$. Polar definitions state that

$$x = r\cos\theta \text{ and } y = r\sin\theta, \text{ or } r = \sqrt{x^2 + y^2} \text{ and } 0 \leq \theta < 2\pi.$$

The sine and cosine functions are periodic, thus $\sin(\theta) = \sin(\theta + 2\pi n)$ for every integer *n*. The same holds for cosine, thus creating an infinite number of ways to represent *z*. By use of the Taylor series, *z* can be re-written in polar form as $z = re^{i(\theta+2\pi n)}$. Let $w = \ln(z)$, then

$$w = \ln(z) = \ln\left(re^{i(\theta+2\pi n)}\right) = \ln(r) + \ln\left(e^{i(\theta+2\pi n)}\right) = \ln(r) + i(\theta + 2\pi n)$$

Therefore, $\ln(z)$ has an infinite number of solutions since *n* can be any integer (Huestis, 1995).

Chapter 3: *Conclusion*

From Archimedes to Napier to Euler, logarithms have taken on many forms and have been used for many mathematical applications. From being developed to improve the ability to multiply large numbers to applying them to two dimensional manifolds (surfaces defined by complex valued functions), logarithms have helped improve the ability to do massive calculations. The connection between logarithms and *e* was revolutionary. Since the number *e* was used so often, many of the modern calculus definitions, such as the derivative, were found to be easier to work with when using the natural logarithm versus logarithms with other bases. The explanation and proofs for these formulas are easily interpreted graphically, algebraically or using a combination of the two in a calculus class. The applications of the logarithm are endless as well. From analyzing population growth, to calculating interest rates and account balances, to measuring the destructive power of earthquakes, to calculating the stellar magnitude of celestial objects, the logarithm has been an integral part of science. At one point logarithms "literally lengthened the life spans of astronomers," and no doubt have done the same for many mathematicians and scientists since (Pierce, 1977, p. 26).

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Vita

Meaghan Whitley Bennett was born in Austin, Texas on July 6, 1983. She was born to Gordon Anderson Bennett and Karen Elaine Whitley Bennett. After graduating from the LBJ Science Academy in Austin, Texas, in 2001, she attended the University of Texas at Austin. In 2002 she joined the UTeach Program and majored in mathematics. She graduated with a Bachelor of Arts in Mathematics with a Secondary Teaching Option in December 2004. She has been teaching mathematics at Stony Point High School since graduation. She entered the Graduate School at the University of Texas at Austin in June 2007.

Permanent address: 3000 Lafayette Ave.
Austin, TX 78722

This report was typed by the author.